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On steady surface waves over a rough periodic bottom — relations between the pattern of imperfect bifurcation and the shape of the bottom

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1 Introduction

In this communication we are concerned with a free boundary problem for two-dimensional steady irrotational flow of incompressible ideal fluid over a periodic bottom. We take the gravity into account as an external force and neglect the effect of surface tension on the free surface. We assume that the domain Ω occupied by the fluid, the free surface Γ and the bottom Σ are of the following forms

$$\Omega = \{z = (z_1, z_2)\ ;\ b(z_1) < z_2 < \eta(z_1),\ z_1 \in \mathbf{R}^1\},$$
$$\Gamma = \{z = (z_1, z_2)\ ;\ z_2 = \eta(z_1),\ z_1 \in \mathbf{R}^1\},$$
$$\Sigma = \{z = (z_1, z_2)\ ;\ z_2 = b(z_1),\ z_1 \in \mathbf{R}^1\},$$

where b is a given function while η is the unknown. The motion of the fluid is described by the velocity $v = (v_1, v_2)$ and the pressure p satisfying the equations

$$\rho(v \cdot \nabla)v + \nabla p = -\rho(0, g) \quad \text{in} \quad \Omega, \tag{1}$$

$$\nabla \cdot v = 0, \quad \nabla^{\perp} \cdot v = 0 \quad \text{in} \quad \Omega, \tag{2}$$

where ρ is a constant density and g is the gravitational constant. The boundary conditions on the free surface Γ are given by

$$p = p_0, \quad v \cdot n_f = 0 \quad \text{on} \quad \Gamma, \tag{3}$$

where p_0 is an atmospheric pressure assumed to be constant and n_f is the unit normal vector to Γ. The boundary condition on the bottom Σ is given by

$$v \cdot n_b = 0 \quad \text{on} \quad \Sigma, \tag{4}$$

where n_b is the unit normal vector to Σ. Moreover, we assume that the motion of the fluid is symmetric with respect to z_2-axis and l-periodic with respect to z_1. Then, we should impose compatibility conditions that the function b is even and l-periodic.

In the case where the function b is identically zero, $\eta(z_1) = \eta_0$, $v(z) = (V, 0)$ and $p(z) = p_0 - \rho g(z_2 - \eta_0)$ satisfy the above system of equations with positive constants η_0 and V, that is to say, uniform flow with flat surface becomes a solution of the system, if the bottom is flat. Even in the case where b is not identically zero, it is natural to expect that there exists a solution (η, v, p) of the system satisfying the condition

$$|v(z) - (V, 0)| \ll 1, \tag{5}$$

if the function b is small in a suitable sense. We will find such solutions. In this problem, data are the bottom Σ and suitable parameters, while the unknowns are the velocity v, the pressure p and the free surface Γ. Therefore, this is a stationary free boundary problem.

There are many results concerning steady surface waves. For example, the existence of two-dimensional periodic stationary surface waves in water of infinite depth was first proved by A. I. Nekrasov [7], [8]. Later, T. Levi-Civita [5] solved independently the same problem by different method. His formulation is very useful to analyze steady surface waves. Then, D. J. Struik [9] extended Levi-Civita's result to the case of presence of a flat bottom. M. A. Lavrentiev [4] considered the problem in the same situation as Struik's and showed the existence of solution by using a variational principle in the theory of conformal and quasiconformal mapping. Moreover, he showed that as the period goes to infinity the corresponding solution converges to a solitary wave. This is the first existence theorem of solitary waves for full surface waves. Then, K. O. Friedrichs & D. H. Hyers [1] gave more direct proof of existence of solitary waves. They adopted Levi-Civita's formulation and treated the problem as a bifurcation problem. All the results mentioned above can be regarded as bifurcation problems and they obtained the bifurcated solution from the trivial solution which is the uniform flow with flat surface. R. Gerber [2] considered the surface waves over a periodically variable bottom and showed the existence of solution by making use of the principle of Leray-Schauder. V. I. Nalimov [6] considered the surface waves over a compactly perturbed non-flat bottom and showed the existence of solution, which has different asymptotic behavior at infinity. Gerber and Nalimov also used Levi-Civita's formulation. However, they did not consider the solutions near the bifurcation point.

Our aim is to analyze the set of all the solutions near the bifurcation point, especially to classify patterns of bifurcation diagram according to the shape of the bottom. Following Levi-Civita we first reformulate the problem assuming (5). Then, introducing the other dependent variables we reformulate it again. Our formulation is slightly useful than Levi-Civita' one. The reduced problem includes three non-dimensional parameters λ the Froud number, β which represents shallowness of the fluid and ε which represents amplitude of the bottom. We will regard λ as a bifurcation parameter. Put $\lambda_n = n/\tanh(n\beta)$ for $n = 1, 2, 3, \ldots$. If $\varepsilon = 0$, that is, the bottom is flat, then we have the pitchfork bifurcation at $\lambda = \lambda_n$ for any $n = 1, 2, 3, \ldots$ and all the bifurcations occur subcritically. Roughly speaking, λ is proportional to V^{-2} so that we obtain many oscillating solutions as the speed of the flow becomes slow. This fact may be familiar. In the case $0 < \varepsilon \ll 1$, the corresponding bifurcation equation is subject to a small perturbation. By Golubitsky-Shaeffer's theory[3] we know a universal unfolding of the pitchfork, which has two unfolding parameters, and hence all possible patterns of the bifurcation diagram. Nine is the number of the patterns including the pitchfork itself: four of them are persistent and the others are nonpersistent diagrams. Therefore, the bifurcation diagram must be equivalent to one of the nine patterns. We want to know which pattern is realized. We will give sufficient conditions in terms of the Fourier coefficients of b under which each persistent bifurcation diagram is realized.

2 Formulation of the problem

Assume that (η, v, p) is a solution of problem (1)–(4), l-periodic with respect to z_1 and satisfy condition (5) and that the motion of the fluid is symmetric with respect to z_2-axis. In terms of v, such a symmetrical property can be written as

$$v_1(-z_1, z_2) = v_1(z_1, z_2), \quad v_2(-z_1, z_2) = -v_2(z_1, z_2). \tag{6}$$

To begin with, we change independent variables. Since Ω is simply connected, (2) implies that there exist single valued stream function ψ and velocity potential φ, which are uniquely determined up to additive constants. Then, the complex velocity potential $\chi = \varphi + i\psi$ is an analytic function of z and it holds that

$$\frac{d\chi}{dz} = \overline{v} = v_1 - iv_2. \tag{7}$$

The kinematical boundary conditions $(3)_2$ and (4) imply that ψ is constant on each boundary Γ and Σ. Therefore, adding a suitable constant to ψ we obtain that $\psi = 0$ on Σ and $\psi = \psi_0$ on Γ with a positive constant ψ_0, where we used (5). Moreover, (5) also implies that a mapping

$\Phi: \Omega \to \mathbf{R}^1 \times (0, \psi_0)$ of the form $\Phi(z) = (\varphi(z), \psi(z))$ is bijective. Therefore, we can regard z as a function of (φ, ψ). Hereafter, we take (φ, ψ) as independent variables and z as dependent variable. Define a constant φ_0 by $\varphi_0 = \int_{z_0}^{z_0+(l,0)} v_1 dz_1 + v_2 dz_2$, which is positive because of (5) and independent of $z_0 \in \Omega$. Then, it holds that

$$(8) \qquad z_1(\varphi + \varphi_0, \psi) = z_1(\varphi, \psi) + l, \quad z_2(\varphi + \varphi_0, \psi) = z_2(\varphi, \psi).$$

By (6), if we add a suitable constant to φ, then we have

$$(9) \qquad z_1(-\varphi, \psi) = -z_1(\varphi, \psi), \quad z_2(-\varphi, \psi) = z_2(\varphi, \psi).$$

Next, we introduce new dependent variables θ and τ by

$$(10) \qquad \overline{v} = \frac{\varphi_0}{l} \exp\{-i(\theta + i\tau)\},$$

more precisely, by $\theta = \arctan(v_2/v_1)$ and $\tau = \log(l|v|/\varphi_0)$. Then, $\theta + i\tau$ is an analytic function of χ and φ_0-periodic with respect to φ because of (8). (9) implies that θ and τ are odd and even functions with respect to φ, respectively. From (7) and (10) we obtain

$$(11) \qquad z(\chi) = z_0 + (\varphi_0/l)^{-1} \int_0^{\chi} \exp\{i(\theta + i\tau)\} d\chi,$$

where $z_0 = (0, z_2(0)) \in \Sigma$. By $(2)_2$, we can rewrite (1) as $\nabla(\frac{1}{2}|v|^2 + \frac{1}{\rho}p + gz_2) = 0$ in Ω. This and the dynamical boundary condition $(3)_1$ imply that $\frac{1}{2}|v|^2 + gz_2 = \text{const.}$ on Γ. This is known as Bernoulli's law. Putting (10) and (11) into this equation and differentiating it with respect to the tangential direction φ yield that $\tau_\varphi + g(\varphi_0/l)^{-3} \mathrm{e}^{-3\tau} \sin\theta = 0$ on $\psi = \psi_0$. This is the boundary condition on the free surface. By (4), (11) and the definition of θ, we see that $\theta = \arctan\{b'(z_1(\varphi, 0))\} = \arctan\{b'((\varphi_0/l)^{-1} \int_0^\varphi \mathrm{e}^{-\tau} \cos\theta d\varphi)\}$ on $\psi = 0$. This is the boundary condition on the bottom. Putting (11) into (8) and using the fact that θ and τ is φ_0-periodic with respect to φ, we see that $\varphi_0^{-1} \int_0^{\varphi_0} \exp\{i(\theta + i\tau)\} d\varphi = 1$

for any $\psi \in [0, \psi_0]$. This is the periodicity condition. Finally, we rewrite the above conditions in the non-dimensional form. To this end, we rescale variables by

$$\chi = \frac{\varphi_0}{2\pi}\tilde{\chi}, \quad \theta(\chi) = \tilde{\theta}(\frac{2\pi}{\varphi_0}\chi), \quad \tau(\chi) = \tilde{\tau}(\frac{2\pi}{\varphi_0}\chi), \quad b(z_1) = h\tilde{b}(\frac{2\pi}{l}z_1)$$

with a positive constant h and introduce non-dimensional parameters by

$$\lambda = \frac{gl}{2\pi(\varphi_0/l)^2}, \qquad \beta = \frac{2\pi\psi_0}{\varphi_0}, \qquad \varepsilon = \frac{2\pi h}{l}.$$

After that, we drop the $\tilde{}$ sign in the notation. Then, the problem is reformulated as follows.

Problem 0 Given $\lambda, \varepsilon \in \mathbf{R}^1$, $\beta > 0$ and 2π-periodic even function b, find functions $\theta + i\tau$ which are analytic in $\{\chi = \varphi + i\psi \,;\ 0 < \psi < \beta,\ \varphi \in \mathbf{R}^1\}$, 2π-periodic with respect to φ and satisfy the conditions

$$\begin{cases} \dfrac{\partial\theta}{\partial\psi} - \lambda \mathrm{e}^{-3\tau}\sin\theta = 0 \qquad \text{on} \qquad \psi = \beta, \\ \theta = \arctan\Big(\varepsilon b'\big(\int_0^{\varphi} \mathrm{e}^{-\tau}\cos\theta d\varphi\big)\Big) \qquad \text{on} \qquad \psi = 0, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \mathrm{e}^{-\tau(\varphi,\psi)}(\cos\theta(\varphi,\psi) + i\sin\theta(\varphi,\psi))d\varphi = 1 \qquad \text{for} \qquad 0 \le \psi \le \beta, \\ \theta(-\varphi,\psi) = -\theta(\varphi,\psi), \quad \tau(-\varphi,\psi) = \tau(\varphi,\psi). \end{cases}$$

Although we can proceed the analysis adopting this formulation, it will be slightly easier to handle the problem if we use another formulation. We introduce another dependent variables u_1 and u_2 by

$$u_1 = \mathrm{e}^{-\tau}\cos\theta - 1, \qquad u_2 = \mathrm{e}^{-\tau}\sin\theta.$$

In terms of $u = u_1 + iu_2$, the problem can be rewritten as follows.

Problem 1 Given $\lambda, \varepsilon \in \mathbf{R}^1$, $\beta > 0$ and 2π-periodic even function b, find functions $u = u_1 + iu_2$ which are analytic in $\{\chi = \varphi + i\psi \ ; \ 0 < \psi < \beta, \ \varphi \in \mathbf{R}^1\}$, 2π-periodic with respect to φ and satisfy the conditions

$$\frac{\partial u_2}{\partial \psi} - \lambda u_2 = F(u) \quad \text{on} \quad \psi = \beta, \tag{12}$$

$$u_2 = \varepsilon G(u; b) \quad \text{on} \quad \psi = 0, \tag{13}$$

$$\int_0^{2\pi} u_1(\varphi, \psi) d\varphi = \int_0^{2\pi} u_2(\varphi, \psi) d\varphi = 0 \quad \text{for} \quad 0 \leq \psi \leq \beta, \tag{14}$$

$$u_1(-\varphi, \psi) = u_1(\varphi, \psi), \quad u_2(-\varphi, \psi) = -u_2(\varphi, \psi), \tag{15}$$

where

$$F(u)(\varphi) = \frac{1}{2}\frac{\partial}{\partial \varphi}\Big(\frac{1}{(u_1(\varphi, \beta) + 1)^2 + u_2(\varphi, \beta)^2} - 1 + 2u_1(\varphi, \beta)\Big), \tag{16}$$

$$G(u; b)(\varphi) = \frac{\partial}{\partial \varphi} b(\varphi + \int_0^{\varphi} u_1(\varphi, 0) d\varphi). \tag{17}$$

In the case $\varepsilon = 0$, $u = 0$ is a solution of the above problem and corresponds to uniform flow with flat surface. We will seek small solutions of Problem 1.

3 Preliminaries

For a non-negative integer s, we denote by H^s the usual Sobolev space of 2π-periodic functions on $\mathbf{R}^1$ equipped with the norm $\|u\|_s = (\Sigma_{n=0}^{s} \int_0^{2\pi} |u^{(n)}(\varphi)|^2 d\varphi)^{1/2}$, where $u^{(n)}$ is the n-th derivative of u. We denote by $\dot{H}^s$ the space of all functions which belong to H^s and have mean value zero. We denote by H^s_{even} and H^s_{odd} the spaces of all even and odd functions in H^s, respectively. We define subspaces $\dot{H}^s_{even}$ and $\dot{H}^s_{odd}$ of $\dot{H}^s$ in a similar way. A pseudo-differential operator $K(D; \psi, \lambda)$, which depends on parameters (ψ, λ) and acts on the space $\dot{H}^s_{odd}$, with a symbol $K(n; \psi, \lambda)$ is defined by $(K(D; \psi, \lambda)u)(\varphi) = \Sigma_{n=1}^{\infty} K(n; \psi, \lambda) u_n \sin n\varphi$ for $u(\varphi) = \Sigma_{n=1}^{\infty} u_n \sin n\varphi$.

We review a notion and several facts in the bifurcation theory. We refer to [3] for further details.

Proposition 1 *$G(x, \lambda, \alpha) = x^3 + \lambda x + \alpha_1 + \alpha_2 x^2$ is a versal unfolding (in fact, universal unfolding) of the pitchfork $g(x, \lambda) = x^3 + \lambda x$, where $\alpha = (\alpha_1, \alpha_2)$ are unfolding parameters, x is the state variable and λ is the bifurcation parameter. More precisely, it holds that for any $H(x, \lambda, \beta) \in C^\infty(\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{R}^n)$ satisfying the condition $H(x, \lambda, 0) = g(x, \lambda)$ there exist a neighbourhood U of $0 \in \mathbf{R}^{n+2}$ and C^∞-mappings S, X, Λ and A such that $H(x, \lambda, \beta) = S(x, \lambda, \beta)G(X(x, \lambda, \beta), \Lambda(\lambda, \beta), A(\beta))$ on U, where $S(x, \lambda, 0) \equiv 1$, $X(x, \lambda, 0) \equiv x$, $\Lambda(\lambda, 0) \equiv \lambda$ and $A(0) = 0$.*

The space $\mathbf{R}^2$ of the unfolding parameters $\alpha = (\alpha_1, \alpha_2)$ is divided into four regions by two curves $\alpha_1 = 0$ and $\alpha_1 = \alpha_2^3/27$. If α is a point on the curves, the corresponding bifurcation diagram is nonpersistent. Otherwise, we obtain a persistent diagram. Moreover, equivalent diagrams are obtained for all α within a given region so that we have four different persistent bifurcation diagrams, which are illustrated in Figure 1.

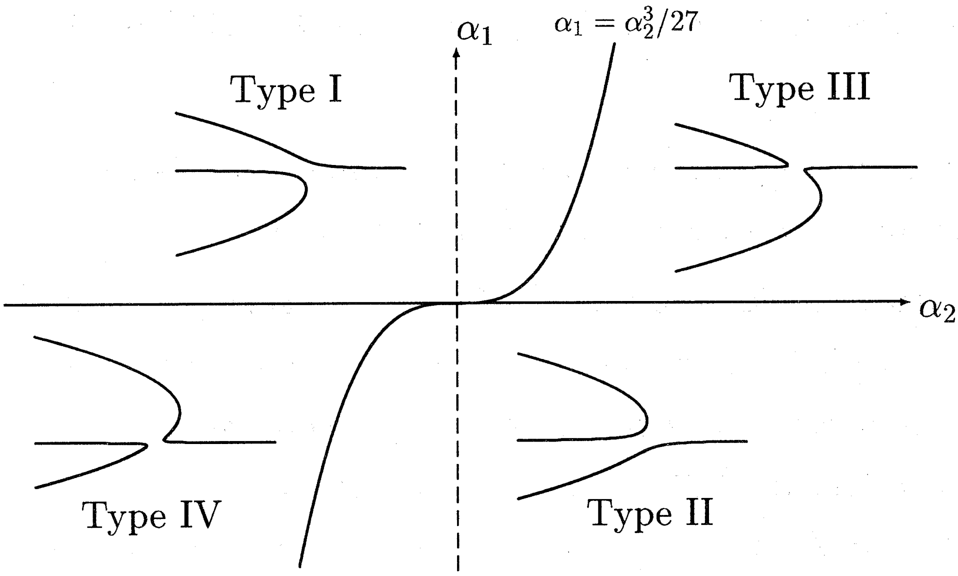

Figure 1: Bifurcation diagrams of $G(x, \lambda, \alpha) = 0$

Here, we have to notice that Proposition 1 gives only a local property. In fact, we do not know the size of the neighbourhood U. This situation may cause a problem. For example, for a fixed $\alpha = (\alpha_1, \alpha_2)$ satisfying $\alpha_1 \neq 0$ the equation $G(x, \lambda, \alpha) = 0$ has no solution in a small neighbourhood of $(x, \lambda) = 0$. In such a case, we can get no information from Proposition 1. This consideration leads us to introduce the following definition.

Definition 1 Let $f(x, \lambda)$ and $g(x, \lambda)$ be C^∞-functions on $U \times L$, where U and L are closed intervals. We say that f and g are globally equivalent on $U \times L$ if there exit a diffeomorphism $\Phi \colon U \times L \to U \times L$ of the form $\Phi(x, \lambda) = (X(x, \lambda), \Lambda(\lambda))$ and C^∞-function $S(x, \lambda)$ such that $g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda))$ on $U \times L$, where $S(x, \lambda)$, $X_x(x, \lambda)$ and $\Lambda'(\lambda)$ are positive on $U \times L$ and Φ maps each face of $\partial(U \times L)$ onto itself.

Definition 2 Let $R > 0$ and $G(x, \lambda, \alpha) = x^3 + \lambda x + \alpha_1 + \alpha_2 x^2$. For a C^∞-function $f(x, \lambda)$, we say that f is a bifurcation of Type I on $[-R, R]^2$ if there exits $\alpha = (\alpha_1, \alpha_2)$ satisfying the conditions $\alpha_1 > 0$ and $\alpha_1 > \alpha_2^3/27$ such that f and $G(\cdot, \cdot, \alpha)$ are globally equivalent on $[-R, R]^2$. In a similar way, we define bifurcations of Type II, Type III and Type IV: see Figure 1.

In order to determine which type of bifurcation is realized, we will make use of a scaling and the following proposition.

Proposition 2 *Suppose that $\alpha = (\alpha_1, \alpha_2)$ satisfies the conditions $\alpha_1 \neq 0$ and $\alpha_1 \neq \alpha_2^3/27$. Then, the bifurcation diagram of $G(x, \lambda, \alpha) \equiv x^3 + \lambda x + \alpha_1 + \alpha_2 x^2 = 0$ is persistent in the following sense: there exists a positive constant $R_0 = R_0(\alpha)$ such that for any $R \geq R_0$ and any $\varphi \in C^\infty([-R, R]^2 \times [-1, 1])$ there exists a small positive constant $\varepsilon_0 = \varepsilon_0(R, \varphi, \alpha)$ such that $G(x, \lambda, \alpha) + \varepsilon\varphi(x, \lambda, \varepsilon)$ and $G(x, \lambda, \alpha)$ are globally equivalent on $[-R, R]^2$ for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

4 Linearized problem

If the function b is even and $u = (u_1, u_2)$ satisfies (15), then $F(u)$ and $G(u; b)$ defined by (16)–(17) are both odd functions. In view of this, we consider the following linearized problem.

Linearized Problem Given $\lambda \in \mathbf{R}^1$, $\beta > 0$, 2π-periodic even function b and odd functions f and g, find functions $u = u_1 + iu_2$ which are analytic in $\{\chi = \varphi + i\psi \ ; \ 0 < \psi < \beta, \ \varphi \in \mathbf{R}^1\}$, 2π-periodic with respect to φ and satisfy the conditions

$$\frac{\partial u_2}{\partial \psi} - \lambda u_2 = f \quad \text{on} \quad \psi = \beta, \tag{18}$$

$$u_2 = g \quad \text{on} \quad \psi = 0, \tag{19}$$

$$\int_0^{2\pi} u_1(\varphi, \psi) d\varphi = \int_0^{2\pi} u_2(\varphi, \psi) d\varphi = 0 \quad \text{for} \quad 0 \leq \psi \leq \beta, \tag{20}$$

$$u_1(-\varphi, \psi) = u_1(\varphi, \psi), \quad u_2(-\varphi, \psi) = -u_2(\varphi, \psi). \tag{21}$$

Define $\lambda_n = \lambda_n(\beta)$ by

$$\lambda_n = \frac{n}{\tanh n\beta}. \tag{22}$$

Then, it holds that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$. Introducing pseudo-differential operators $K_j(D; \psi, \lambda)$, $j = 1, 2, 3, 4$, depending on parameters (ψ, λ) by

$$\begin{cases} K_1(n; \psi, \lambda) = \dfrac{n \sinh n(\beta - \psi) - \lambda \cosh n(\beta - \psi)}{n \cosh n\beta - \lambda \sinh n\beta}, \\[2ex] K_2(n; \psi, \lambda) = -\dfrac{\cosh n\psi}{n \cosh n\beta - \lambda \sinh n\beta}, \\[2ex] K_3(n; \psi, \lambda) = \dfrac{n \cosh n(\beta - \psi) - \lambda \sinh n(\beta - \psi)}{n \cosh n\beta - \lambda \sinh n\beta}, \\[2ex] K_4(n; \psi, \lambda) = \dfrac{\sinh n\psi}{n \cosh n\beta - \lambda \sinh n\beta}, \end{cases} \tag{23}$$

we have the following lemma.

Lemma 1 *Let $\beta > 0$ and $\lambda \in \mathbf{R}^1$. Suppose that $s \geq 1$ is an integer, $f \in H^{s-1}_{odd}$ and $g \in H^s_{odd}$.*

(i) *If $\lambda \neq \lambda_n$ for all $n = 1, 2, 3, \ldots$, then the Linearized Problem has a unique solution $u \in C([0,\beta]; H^s)$ of the form*

$$(24) \qquad \begin{cases} u_1(\cdot, \psi) = T(K_1(D; \psi, \lambda)g + K_2(D; \psi, \lambda)f), \\ u_2(\cdot, \psi) = K_3(D; \psi, \lambda)g + K_4(D; \psi, \lambda)f, \end{cases}$$

where T is a linear operator defined by

$$(25) \qquad (Tu)(\varphi) = \sum_{n=1}^{\infty} u_n \cos n\varphi \quad \text{for} \quad u(\varphi) = \sum_{n=1}^{\infty} u_n \sin n\varphi.$$

(ii) *If $\lambda = \lambda_m$ for some m, then the Linearized Problem has a solution if and only if the compatibility condition*

$$(26) \qquad (m \sinh m\beta - \lambda_m \cosh m\beta) g_m = f_m$$

is satisfied, where f_m and g_m are the m-th Fourier coefficients of f and g, respectively. In this case, the set of all the solutions forms a one-dimensional affine space.

In the case $\lambda \neq \lambda_n$ for all $n = 1, 2, 3, \ldots$, by making use of this lemma and standard iteration arguments we can show the following theorem.

Theorem 1 *Suppose that $\varepsilon = 1$, $\beta > 0$ and $\inf_{n \geq 1} |1 - \lambda/\lambda_n| \geq \delta > 0$. There exists a small positive constant ε_0 depending only on β and δ such that if s and m are integers satisfying $m \geq s \geq 1$ and*

$$(27) \qquad b \in H^{s+1}_{even}, \quad \|b\|_{s+1} \leq M, \quad \|b\|_2 \leq \varepsilon_0,$$

then the Problem 1 has a solution $u \in C([0,\beta]; H^s)$ satisfying

$$(28) \qquad \sup_{0 \leq \psi \leq \beta} \|u(\cdot, \psi)\|_s \leq C_1 \|b\|_{s+1}, \quad \|u(\cdot, \beta)\|_m \leq C_2 \|b\|_2,$$

where $C_1 = C_1(\beta, \delta, s, M)$ and $C_2 = C_2(\beta, \delta, m)$ are positive constants.

Remarks (i) Uniqueness of solutions does not hold in general.

(ii) Since m is arbitrary, we see that $u(\cdot, \beta)$ is a C^∞-function. This fact implies that the free surface Γ is a C^∞-curve even if the bottom Σ is not smooth.

(iii) We have the same result even if we do not assume the symmetry of the motion of the fluid.

5 Bifurcation equation

In the following, we fix a positive integer m and consider the case where λ is in a neighbourhood of λ_m. For simplicity, we also assume that the function b is of C^∞-class. Define mappings $H(u, \lambda, \varepsilon)$ and $E(u, \lambda, \varepsilon)$ by

$$(29)\quad \begin{cases} H(u,\lambda,\varepsilon) = (H_1(u,\lambda,\varepsilon), H_2(u,\lambda,\varepsilon)), \\ H_1(u,\lambda,\varepsilon)(\cdot,\psi) \\ \quad = T(\varepsilon K_1(D;\psi,0)G(u;b) + K_2(D;\psi,0)(\lambda u_2(\cdot,\beta) + F(u))), \\ H_2(u,\lambda,\varepsilon)(\cdot,\psi) \\ \quad = \varepsilon K_3(D;\psi,0)G(u;b) + K_4(D;\psi,0)(\lambda u_2(\cdot,\beta) + F(u)), \end{cases}$$

and

$$(30)\qquad E(u,\lambda,\varepsilon) = u - H(u,\lambda,\varepsilon),$$

respectively. Then, by Lemma 1 for $\lambda = 0$ Problem 1 is reduced to finding zero points of $E(\cdot, \lambda, \varepsilon)$.

Next, we reduce the problem to a finite-dimensional one by adopting the so-called Lyapunov-Schmidt reduction. To this end, we define a fundamental function space X by

$$(31)\quad X = \{u = (u_1, u_2) \in C([0,\beta]; \dot{H}^1_{even} \times \dot{H}^1_{odd})\ ; \\ u_1 + iu_2 \text{ is analytic in } \chi = \varphi + i\psi,\ 0 < \psi < \beta,\ \varphi \in \mathbf{R}^1\}.$$

Then, we see that

$$(32)\qquad \mathrm{Ker}(E_u(0,\lambda,0)) = \{0\},\ \ \mathrm{Range}(E_u(0,\lambda,0)) = X$$

if $\lambda \neq \lambda_n$ for all $n = 1, 2, 3, \ldots$ and that

$$(33) \qquad \begin{cases} \mathrm{Ker}(E_u(0, \lambda_m, 0)) = \{x\xi_m \,;\, x \in \mathbf{R}^1\}, \\ \mathrm{Range}(E_u(0, \lambda_m, 0)) \oplus \mathrm{Ker}(E_u(0, \lambda_m, 0)) = X \end{cases}$$

for $m = 1, 2, 3, \ldots$, where

$$(34) \qquad \begin{aligned} \xi_m(\varphi, \psi) &= (K_2(m; \psi, 0) \cos m\varphi,\ K_4(m; \psi, 0) \sin m\varphi) \\ &= \Big(-\frac{\cosh m\psi}{m \cosh m\beta} \cos m\varphi,\ \frac{\sinh m\psi}{m \cosh m\beta} \sin m\varphi\Big). \end{aligned}$$

Using (33) and the Lyapunov-Schmidt reduction, we obtain a bifurcation equation $f(x, \lambda, \varepsilon) = 0$, where f is a scalar function of C^∞-class in a neighbourhood of $(x, \lambda, \varepsilon) = (0, \lambda_m, 0)$, x is a state variable, λ is a bifurcation parameter and ε is a small parameter.

We introduce symbols $\widetilde{K}_j(D; \psi, \lambda)$, $j = 1, 2, 3, 4$, by

$$(35) \qquad \widetilde{K}_j(n; \psi, \lambda) = \begin{cases} K_j(n; \psi, \lambda) & \text{if } n \geq 1,\ n \neq m, \\ K_j(n; \psi, 0) & \text{if } n = m \end{cases}$$

for $j = 1, 3$ and

$$(36) \qquad \widetilde{K}_j(n; \psi, \lambda) = \begin{cases} K_j(n; \psi, \lambda) & \text{if } n \geq 1,\ n \neq m, \\ 0 & \text{if } n = m \end{cases}$$

for $j = 2, 4$. For notational convenience, we extend $\widetilde{K}_j(n; \psi, \lambda)$ for non-positive n as

$$(37) \quad \begin{cases} \widetilde{K}_j(0; \psi, \lambda) = 0 \quad \text{for} \quad j = 1, 2, 3, 4, \\ \widetilde{K}_j(-n; \psi, \lambda) = -\widetilde{K}_j(n; \psi, \lambda) \quad \text{for} \quad n = 1, 2, 3, \ldots, \quad j = 1, 2, \\ \widetilde{K}_j(-n; \psi, \lambda) = \widetilde{K}_j(n; \psi, \lambda) \quad \text{for} \quad n = 1, 2, 3, \ldots, \quad j = 3, 4. \end{cases}$$

Since we have been assuming that the function b is even and 2π-periodic, it can be expanded as the cosine Fourier series

$$(38) \qquad b(\varphi) = \sum_{n=0}^{\infty} b_n \cos n\varphi.$$

Putting $\tilde{b}_n = -nb_n$, we have $b'(\varphi) = \Sigma_{n=1}^{\infty} \tilde{b}_n \sin n\varphi$. Then, we can expand the bifurcation equation as

$$
\begin{aligned}
(39)\quad f(x,\lambda,\varepsilon) &= \Big(\frac{\lambda}{\lambda_m} - 1\Big)x + C_{30}(\lambda)x^3 + C_{01}(\lambda)\varepsilon + C_{11}(\lambda)\varepsilon x \\
&\quad + C_{21}(\lambda)\varepsilon x^2 + C_{02}(\lambda)\varepsilon^2 + C_{12}(\lambda)\varepsilon^2 x + C_{03}(\lambda)\varepsilon^3 + O(x^4+\varepsilon^4),
\end{aligned}
$$

where

$$
\begin{aligned}
(40)\qquad C_{30}(\lambda) &= \frac{m^2}{4}(3(K_2(m;\beta,0))^2 + (K_4(m;\beta,0))^2) \\
&\quad \times (3K_2(m;\beta,0)K_2(2m;\beta,\lambda) - K_4(m;\beta,0)K_4(2m;\beta,\lambda)) \\
&\quad + \frac{m}{2}K_2(m;\beta,0)(3(K_2(m;\beta,0))^2 - (K_4(m;\beta,0))^2),
\end{aligned}
$$

$$
(41)\qquad C_{01}(\lambda) = \lambda K_3(m;\beta,0)\tilde{b}_m = \frac{\lambda \tilde{b}_m}{\cosh m\beta},
$$

$$
\begin{aligned}
(42)\quad C_{21}(\lambda) &= \frac{1}{8}\lambda K_3(m;\beta,0)\Big\{(K_2(m;0,0))^2(3\tilde{b}_{3m} - \tilde{b}_m) \\
&\qquad + m(3(K_2(m;\beta,0))^2 + (K_4(m;\beta,0))^2)K_2(2m;0,\lambda)(\tilde{b}_{3m} + \tilde{b}_m)\Big\} \\
&\quad + \frac{m}{2}K_2(m;0,0)(3K_2(m;\beta,0)K_1(2m;\beta,\lambda) \\
&\qquad\qquad - K_4(m;\beta,0)K_3(2m;\beta,\lambda))(\tilde{b}_{3m} - \tilde{b}_m) \\
&\quad + \frac{m^2}{2}(3K_2(m;\beta,0)K_2(2m;\beta,\lambda) - K_4(m;\beta,0)K_4(2m;\beta,\lambda)) \\
&\qquad \times \Big\{(3K_2(m;\beta,0)K_1(3m;\beta,\lambda) - K_4(m;\beta,0)K_3(3m;\beta,\lambda))\tilde{b}_{3m} \\
&\qquad\quad + (3K_2(m;\beta,0)K_1(m;\beta,0) + K_4(m;\beta,0)K_3(m;\beta,0))\tilde{b}_m\Big\} \\
&\quad + \frac{m^2}{4}(3(K_2(m;\beta,0))^2 + (K_4(m;\beta,0))^2) \\
&\qquad \times \Big\{(3K_1(3m;\beta,\lambda)K_2(2m;\beta,\lambda) - K_3(3m;\beta,\lambda)K_4(2m;\beta,\lambda))\tilde{b}_{3m} \\
&\qquad\quad + (3K_1(m;\beta,0)K_2(2m;\beta,\lambda) - K_3(m;\beta,0)K_4(2m;\beta,\lambda))\tilde{b}_m\Big\} \\
&\quad - mK_2(m;\beta,0)K_4(m;\beta,0)(K_3(3m;\beta,\lambda)\tilde{b}_{3m} + K_3(m;\beta,0)\tilde{b}_m) \\
&\quad + \frac{m}{2}(9(K_2(m;\beta,0))^2 - (K_4(m;\beta,0))^2)K_1(m;\beta,0)\tilde{b}_m \\
&\quad + \frac{m}{2}(3(K_2(m;\beta,0))^2 + (K_4(m;\beta,0))^2)K_1(3m;\beta,\lambda)\tilde{b}_{3m},
\end{aligned}
$$

$$(43)\ C_{02}(\lambda) = \frac{m}{2}\lambda K_3(m;\beta,0)\Big(\sum_{n+l=m} - \sum_{n-l=\pm m}\Big)\frac{1}{n}\tilde{b}_n\tilde{b}_l\widetilde{K}_1(n;0,\lambda)$$
$$-\frac{m}{4}\sum_{n+l=m}\tilde{b}_n\tilde{b}_l(\widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda)+3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda))$$
$$+\frac{m}{4}\sum_{n-l=\pm m}\tilde{b}_n\tilde{b}_l(\widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda)-3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda)),$$

$$(44)\ C_{03}(\lambda) = \lambda K_3(m;\beta,0)\frac{m}{8}\Big[$$
$$\Big(\sum_{n+l+k=m} - \sum_{n+l-k=\pm m}\Big)\tilde{b}_n\tilde{b}_l\tilde{b}_k\Big\{\frac{2}{n}\widetilde{K}_1(n;0,\lambda)\widetilde{K}_1(n+l;0,\lambda)$$
$$-(3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda)+\widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda))\widetilde{K}_2(n+l;0,\lambda)\Big\}$$
$$+\Big(\sum_{n-l-k=\pm m} - \sum_{n-l+k=\pm m}\Big)\tilde{b}_n\tilde{b}_l\tilde{b}_k\Big\{\frac{2}{n}\widetilde{K}_1(n;0,\lambda)\widetilde{K}_1(n-l;0,\lambda)$$
$$+(3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda)-\widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda))\widetilde{K}_2(n-l;0,\lambda)\Big\}$$
$$+\Big(\sum_{n+l+k=m} + \sum_{n+l-k=\pm m}\Big)\frac{k}{nl}\tilde{b}_n\tilde{b}_l\tilde{b}_k\widetilde{K}_1(n;0,\lambda)\widetilde{K}_1(l;0,\lambda)$$
$$-\Big(\sum_{n-l+k=\pm m} + \sum_{n-l-k=\pm m}\Big)\frac{k}{nl}\tilde{b}_n\tilde{b}_l\tilde{b}_k\widetilde{K}_1(n;0,\lambda)\widetilde{K}_1(l;0,\lambda)\Big]$$
$$+\frac{m}{4}\Big\{-\sum_{n+l+k=m}\frac{n+l}{n}\tilde{b}_n\tilde{b}_l\tilde{b}_k\widetilde{K}_1(n;0,\lambda)$$
$$\times(3\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_1(n+l;\beta,\lambda)+\widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_3(n+l;\beta,\lambda))$$
$$-\sum_{n+l-k=\pm m}\frac{n+l}{n}\tilde{b}_n\tilde{b}_l\tilde{b}_k\widetilde{K}_1(n;0,\lambda)$$
$$\times(3\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_1(n+l;\beta,\lambda)-\widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_3(n+l;\beta,\lambda))$$
$$+\sum_{n-l+k=\pm m}\frac{n-l}{n}\tilde{b}_n\tilde{b}_l\tilde{b}_k\widetilde{K}_1(n;0,\lambda)$$
$$\times(3\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_1(n-l;\beta,\lambda)+\widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_3(n-l;\beta,\lambda))$$
$$+\sum_{n-l-k=\pm m}\frac{n-l}{n}\tilde{b}_n\tilde{b}_l\tilde{b}_k\widetilde{K}_1(n;0,\lambda)$$
$$\times(3\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_1(n-l;\beta,\lambda)-\widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_3(n-l;\beta,\lambda))\Big\}$$
$$+\frac{3m}{8}\Big\{\sum_{n+l+k=}m(n+l)\tilde{b}_n\tilde{b}_l\tilde{b}_k$$
$$\times(3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda)+\widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda))$$

$$
\begin{aligned}
&\qquad \times (\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_2(n+l;\beta,\lambda) + \widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_4(n+l;\beta,\lambda)) \\
&+ \sum_{n+l-k=\pm m} (n+l)\tilde{b}_n\tilde{b}_l\tilde{b}_k \\
&\qquad \times (3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda) + \widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda)) \\
&\qquad \times (\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_2(n+l;\beta,\lambda) - \widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_4(n+l;\beta,\lambda)) \\
&+ \sum_{n-l+k=\pm m} (n-l)\tilde{b}_n\tilde{b}_l\tilde{b}_k \\
&\qquad \times (3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda) - \widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda)) \\
&\qquad \times (\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_2(n-l;\beta,\lambda) + \widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_4(n-l;\beta,\lambda)) \\
&+ \sum_{n-l-k=\pm m} (n-l)\tilde{b}_n\tilde{b}_l\tilde{b}_k \\
&\qquad \times (3\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda) - \widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda)) \\
&\qquad \times (\widetilde{K}_1(k;\beta,\lambda)\widetilde{K}_2(n-l;\beta,\lambda) - \widetilde{K}_3(k;\beta,\lambda)\widetilde{K}_4(n-l;\beta,\lambda))\Big\} \\
&+ \frac{m}{2}\Big\{\Big(\sum_{n+l+k=m} + \sum_{n+l-k=\pm m}\Big)\tilde{b}_n\tilde{b}_l\tilde{b}_k \\
&\qquad \times (\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda) + \widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda))\widetilde{K}_1(k;\beta,\lambda) \\
&+ \Big(\sum_{n-l+k=\pm m} + \sum_{n-l-k=\pm m}\Big)\tilde{b}_n\tilde{b}_l\tilde{b}_k \\
&\qquad \times (\widetilde{K}_1(n;\beta,\lambda)\widetilde{K}_1(l;\beta,\lambda) - \widetilde{K}_3(n;\beta,\lambda)\widetilde{K}_3(l;\beta,\lambda))\widetilde{K}_1(k;\beta,\lambda)\Big\}.
\end{aligned}
$$

where $\sum_{n+l=m}$ means the summation of all positive integers n and l satisfying $n+l=m$, etc.

6 Classification of bifurcation diagrams

Now, we are ready to give our main results. First of all, we consider the case $\varepsilon = 0$, that is, the case where the bottom is flat.

Proposition 3 *If $\varepsilon = 0$, then we have pitchfork bifurcations at $(x,\lambda) = (0,\lambda_m)$ for all $m = 1,2,3,\ldots$, and all the bifurcations occur subcritically.*

Proof. It follows from (39) that

$$f(x,\lambda,0) = \Big(\frac{\lambda}{\lambda_m} - 1\Big)x + C_{30}(\lambda)x^3 + O(x^4). \tag{45}$$

Therefore, it is sufficient to show that $C_{30}(\lambda_m)$ is positive for all $m = 1, 2, 3, \ldots$. By (40) and (23), we see that

$$\begin{aligned} C_{30}(\lambda_m) = \frac{1}{4m(\lambda_{2m}-\lambda_m)\sinh 2m\beta}\Big\{ & 6\Big(\frac{\tanh 2m\beta}{\tanh m\beta} - 1\Big)\cosh 2m\beta \\ & + (5\tanh m\beta + \tanh^3 m\beta)(\cosh 2m\beta - \sinh 2m\beta) \\ & + \Big(5\Big(\tanh m\beta - \frac{1}{2}\Big)^2 + \frac{7}{4} + \tanh^2 m\beta(2-\tanh m\beta)\Big)\cosh 2m\beta\Big\}. \end{aligned} \tag{46}$$

This shows the assertion. The proof is complete.

Next, we fix a positive integer m and consider the bifurcation problem near $(x,\lambda) = (0,\lambda_m)$ in the case $0 < \varepsilon \ll 1$. Let us change the state variable x and the bifurcation parameter λ into y and μ in the formula $x = \varepsilon^{1/3}y$ and $\lambda = \lambda_m(1+\varepsilon^{2/3}\mu)$, respectively. Then, we have

$$f(x,\lambda,\varepsilon) = \varepsilon\Big(C_{30}(\lambda_m)y^3 + \mu y + \frac{\lambda_m}{\cosh m\beta}\tilde{b}_m + O(\varepsilon^{1/3})\Big). \tag{47}$$

Therefore, noting that $\tilde{b}_m = -mb_m$ we can apply Proposition 2 to obtain the following theorem.

Theorem 2 *There exists a positive constant ε_0 depending on β, m and b such that for ε satisfying $0 < \varepsilon \le \varepsilon_0$,*

(i) *we have a bifurcation of Type I if $b_m < 0$;*

(ii) *we have a bifurcation of Type II if $b_m > 0$.*

We proceed to consider the case where $b_m = 0$ and $C_{02}(\lambda) \equiv 0$. Let us change the variables x and λ to y and μ in the formula

$$\begin{cases} x = \varepsilon^{2/3}y, \\ \lambda = \lambda_m\Big(1 - C_{11}(\lambda_m)\varepsilon \\ \qquad + (C_{30}(\lambda_m)\mu - C_{12}(\lambda_m) + \lambda_m C_{11}(\lambda_m)C'_{11}(\lambda_m))\varepsilon^2\Big), \end{cases} \tag{48}$$

respectively. Then, we have

$$(49)\quad f(x,\lambda,\varepsilon)=\varepsilon^3 C_{30}(\lambda_m)\Big(y^3+\mu y+\frac{C_{21}(\lambda_m)}{C_{30}(\lambda_m)}y^2+\frac{C_{03}(\lambda_m)}{C_{30}(\lambda_m)}+O(\varepsilon)\Big).$$

Therefore, by Proposition 2 we obtain the following theorem.

Theorem 3 *Suppose that $b_m = 0$ and $C_{02}(\lambda)\equiv 0$. There exists a positive constant ε_0 depending on β, m and b such that for ε satisfying $0<\varepsilon\le\varepsilon_0$,*

(i) *we have a bifurcation of Type I if $A_1 > 0$ and $A_1 > A_2^3/27$;*

(ii) *we have a bifurcation of Type II if $A_1 < 0$ and $A_1 < A_2^3/27$;*

(iii) *we have a bifurcation of Type III if $0 < A_1 < A_2^3/27$;*

(iv) *we have a bifurcation of Type IV if $A_2^3/27 < A_1 < 0$,*

where

$$(50)\quad A_1=A_1(m,\beta,b)=\frac{C_{03}(\lambda_m)}{C_{30}(\lambda_m)},\qquad A_2=A_2(m,\beta,b)=\frac{C_{21}(\lambda_m)}{C_{30}(\lambda_m)}.$$

7 Particular cases

In this section, we apply Theorem 3 to particular cases where the function b has the form

$$(51)\qquad b(\varphi)=b_{3m}\cos 3m\varphi+b_{5m}\cos 5m\varphi$$

or

$$(52)\qquad b(\varphi)=b_{3m}\cos 3m\varphi+b_{7m}\cos 7m\varphi.$$

In both cases, the assumptions of Theorem 3 are satisfied. Moreover, by (42) we see that

$$(53)\qquad C_{21}(\lambda_m)=\widetilde{C}_{21}(m,\beta)b_{3m},$$

where $\widetilde{C}_{21}(m,\beta)$ is a constant depending only on m and β. Putting

$$(54)\qquad \beta_3^*(m)=\frac{1}{m}\log(\sqrt{3/4+1}+\sqrt{3/4}),$$

we see that

$$(55)\qquad \widetilde{C}_{21}(m,\beta)\begin{cases} <0 & \text{for } 0<\beta<\beta_3^*(m), \\ =0 & \text{for } \beta=\beta_3^*(m), \\ >0 & \text{for } \beta_3^*(m)<\beta<\infty. \end{cases}$$

Next, we restrict ourselves to the case (51). By (44) we see that

$$(56)\qquad C_{03}(\lambda_m)=\widetilde{C}_{03}(m,\beta)(b_{3m})^2 b_{5m},$$

where $\widetilde{C}_{03}(m,\beta)$ is a constant depending only on m and β. Moreover, there exist $x_1 \in (0,1/8)$ and $x_2 \in (1/8,1/4)$ such that

$$(57)\qquad \widetilde{C}_{03}(m,\beta)\begin{cases} <0 & \text{for } 0<\beta<\beta_1^*(m) \text{ or } \beta>\beta_2^*(m), \\ =0 & \text{for } \beta=\beta_1^*(m) \text{ or } \beta=\beta_2^*(m), \\ >0 & \text{for } \beta_1^*(m)<\beta<\beta_2^*(m), \end{cases}$$

where

$$(58)\qquad \beta_j^*(m)=\frac{1}{m}\log(\sqrt{x_j+1}+\sqrt{x_j}),\quad j=1,2.$$

Using these facts and Theorem 3 and introducing a constant C by

$$(59)\qquad C=C(m,\beta)=\frac{\widetilde{C}_{21}(m,\beta)}{27(C_{30}(\lambda_m))^2\widetilde{C}_{03}(m,\beta)},$$

we obtain the following tables.

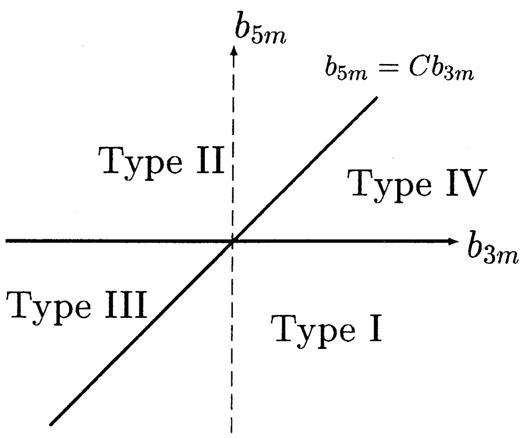

Figure 2: $0<\beta<\beta_1^*(m)$ or $\beta_2^*(m)<\beta<\beta_3^*(m)$

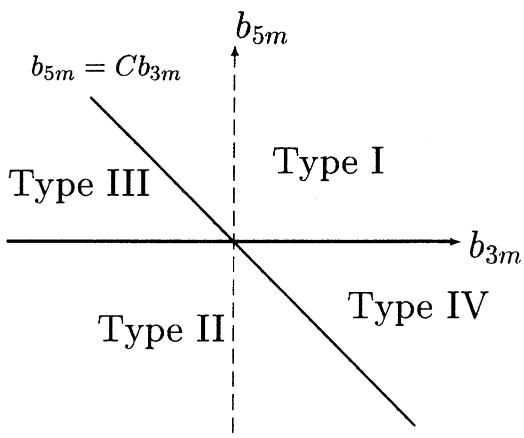

Figure 3: $\beta_1^*(m)<\beta<\beta_2^*(m)$

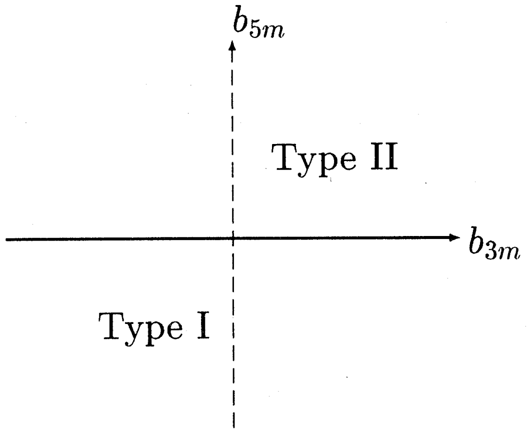

Figure 4: $\beta = \beta_3^*(m)$

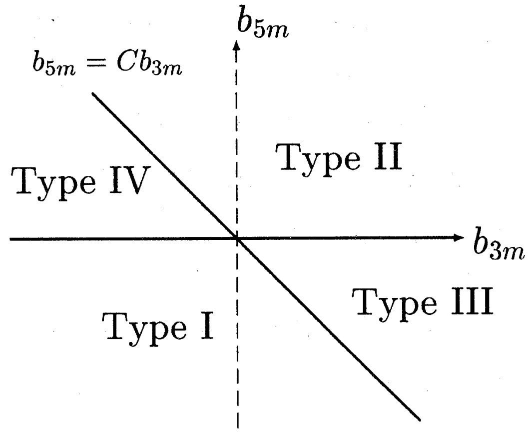

Figure 5: $\beta > \beta_3^*(m)$

Namely, when $0 < \beta < \beta_1^*(m)$ we have a bifurcation of Type III if $C(m,\beta)b_{3m} < b_{5m} < 0$, etc.

Next, we consider the case (52). In this case, we obtain in place of (56) that

$$C_{03}(\lambda_m) = \widetilde{C}_{03}(m,\beta)(b_{3m})^2 b_{7m}, \tag{60}$$

where $\widetilde{C}_{03}(m,\beta)$ is a constant depending only on m and β. Moreover, there exist $x_1 \in (0,1/5)$ and $x_2 \in (1/5,2/5)$ such that (57) holds. Hence, in this case we have almost the same result as the previous one.

The details will be published elsewhere.

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